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# FUNCTIONS OF LIMITED VARIATION AND LEBESGUE INTEGRALS.

BY GOLDIE PRINTIS HORTON.

1. **Introduction.** A fundamental theorem in the theory of Lebesgue integrals is that the four derivates of a function continuous and of limited variation are summable and all equal except over a set of measure zero. This theorem is proved by Lebesgue\* by actually calculating the variation of the function and by Vallée Poussin† by using two auxiliary functions, called majorating and minorating functions, which, with their derivatives, satisfy certain conditions of inequality.

The purpose of this paper is to show how the proof of this theorem can be simplified by the study of a very simple monotone function, which we shall name *measure function*, preliminary to proving directly Lebesgue's theorem that a continuous function with a bounded derivate has a derivative except for a set of measure zero, and to the deduction of known existence theorems for derivatives independently of the theory of majorating functions.

2. **Definitions and known theorems.** For the convenience of the reader we state the following definitions:

DEFINITION I. Given a set of points  $E$  bounded and contained in an interval  $(a, b)$ . Enclose the points of  $E$  in a set of intervals. Let  $\Sigma\alpha$  denote their sum. By definition, the exterior measure of  $E$ , denoted by  $m_e E$ , is the lower limit of all the sums  $\Sigma\alpha$  possible; the interior measure of  $E$ , denoted by  $m_i E$ , is  $(b - a) - m_e CE$ ;‡ and in case  $m_e E = m_i E$ , the set  $E$  is said to be measurable, and its measure,  $mE$ , is the common value of  $m_e E$  and  $m_i E$ .§

DEFINITION II. A function  $f(x)$ , one-valued in a measurable set  $E$  and of determinate sign if infinite, is said to be measurable in  $E$  if at least one of the sub-sets of  $E$  for which  $f \geq A$ ,  $f < A$ ,  $f > A$ ,  $f \leq A$  is measurable whatever constant  $A$  may be.|| It follows then that the other three are also measurable.

\* Leçons sur l'Integration, p. 121.

† Cours d'Analyse, vol. 1, 3d ed., § 261.

‡  $CE$  denotes the complement of  $E$  with respect to  $(a, b)$ , that is, the points of  $(a, b)$  not in  $E$ .

§ If  $mE = 0$ ,  $E$  is called a *nul* set or a set of measure zero.

|| Of the properties of measurable functions attention is called to the facts that the limit or the limits of indetermination of a sequence of measurable functions is measurable, that any continuous function is measurable, and hence the derivates (the right- and left-hand limits of indetermination of the incremental ratio) of a continuous function are measurable. For proofs

DEFINITION III. Let the function  $f(x)$  be single-valued, measurable, and with bounds  $\mu$  and  $M$  in a measurable set  $E$ . Divide its interval of variation  $(\mu, M)$  into consecutive parts by the series of increasing numbers  $l_1 = \mu, l_2, l_3, \dots, l_{n+1} = M$ , and let  $\eta_i$  be a value of  $f(x)$  anywhere on the interval  $(l_i, l_{i+1})$ . Let  $e_i$  denote the measure of the sub-set  $e_i$  of  $E$  where  $l_i \leq f(x) < l_{i+1}$ . It can be proved that  $\lim_{n \rightarrow \infty} \sum_{i=1}^n \eta_i e_i$  exists, and by definition this limit is the Lebesgue integral of  $f(x)$  in the set  $E$  and is written  $L_E f(x) dx$ , or  $\int_E f(x) dx$  in case there is no confusion with the Riemann integral.

In case  $f(x)$  is not bounded, and not negative in  $E$ , let  $f_n = f(x)$  at points of  $E$  where  $f(x) \leq n$ , and  $f_n = n$  where  $f(x) > n$ . Then by definition  $\int_E f(x) dx = \lim_{n \rightarrow \infty} \int_E f_n dx$ , and if this limit is finite  $f(x)$  is said to be *summable* in  $E$ .

In the general case, a non-bounded function  $f(x)$  is the difference of two non-negative functions, and  $f(x)$  is *summable* if these two functions are summable.\*

DEFINITION IV. Let  $f(x)$  be bounded in an interval  $(a, b)$ ,  $a < b$ . Let  $(\alpha, \beta)$  be a sub-interval of  $(a, b)$  such that  $a \leq \alpha < \beta \leq b$ , and call  $f(\beta) - f(\alpha)$  the variation of  $f(x)$  in  $(\alpha, \beta)$ . Consider a set  $(\alpha_i, \beta_i)$  of distinct  $(\alpha, \beta)$  intervals. If for every  $\delta$ , however small, an  $\epsilon$  can be found such that for every set  $(\alpha_i, \beta_i)$  of intervals  $\Sigma[f(\beta_i) - f(\alpha_i)] < \delta$  when  $\Sigma(\alpha_i, \beta_i) < \epsilon$ ,  $f(x)$  is said to be *absolutely continuous* in  $(a, b)$ .†

Reference will be made to the following known theorems:

(A). THEOREM ON CONVERGENCE. If a sequence  $f_1, f_2, \dots, f_n, \dots$  of measurable functions converge toward a finite limit  $f(x)$  in a measurable set  $E$ , for every  $\epsilon$  and  $\delta$ , however small, a value  $N$  of  $n$  can be found such that  $|f(x) - f_n(x)| < \epsilon$  except in a set of measure  $< \delta$  for all values of  $n > N$ .‡

of these properties see Lebesgue's *Leçons sur l'Intégration*, Vallée Poussin's *Cours d'Analyse*, Vallée Poussin's *Intégrales de Lebesgue*, Bliss's *Integrals of Lebesgue*, Bull. Amer. Math. Soc., vol. 24 (1917), pp. 1-46.

\* It is to be noted that a set of measure zero can be neglected in the calculation of an integral, and that it then follows from the definition that a function summable in a set  $E$  can become infinite in a sub-set of  $E$  of measure zero. It can be proved that a function measurable and bounded is summable; and hence if one derivate of a continuous function is bounded all the derivatives are bounded and summable. It follows from the definition that if a function is summable its absolute value is summable and inversely.

† Vitali, "Sulle funzioni integrali," in *Atti della R. Accademia delle Scienze di Torino*, 1905. The statement given here is that of M. B. Porter in "Concerning Absolutely Continuous Functions," Bull. Am. Math. Soc., vol. 22 (1915), pp. 109-111.

‡ For a proof of this theorem see Vallée Poussin, *Cours d'Analyse*, p. 71.

## (B). DISTRIBUTIVE PROPERTIES OF SUMMABLE FUNCTIONS.

$$\sum_1^\infty \int_{E_i} f(x) dx = \int_{\sum_1^\infty E_i} f(x) dx$$

if the  $E_i$  have no points in common, and  $\sum_1^n \int_E f_i(x) dx = \int_E \sum_1^n f_i(x) dx$  if the  $f_i$  are finite in number.\*

(C). THEOREM OF THE MEAN. If  $f(x)$  is summable in a set  $E$ , and  $\mu \leq f(x) \leq M$ ,  $\mu$  and  $M$  finite,  $\mu \text{ meas } E \leq \int_E f(x) dx \leq M \text{ meas } E$ .†

(D). LEBESGUE'S LIMIT THEOREM. If  $\lim_{n \rightarrow \infty} f_n(x) = f(x)$  and  $f_n(x)$  is bounded and measurable in  $E$ ,  $\lim_{n \rightarrow \infty} \int_E f_n(x) dx = \int_E f(x) dx$ .‡

(E). THEOREM. A function continuous in an interval cannot have a derivate as  $\Lambda$  negative (positive) only in a set of measure zero and have that derivate finite in that set.§

For later reference we prove the following

LEMMA. If  $f(x)$  is continuous in  $(a, b)$

$$f(x) - f(a) \equiv V_a^x f(x) = \lim_{h \rightarrow 0} \int_a^{x+h} \frac{f(x) - f(x-h)}{h} dx, \quad x < b,$$

uniformly for all values of  $x$  in  $(a, b)$ , where  $\int$  denotes the Riemann integral.||

Consider the identity

$$\begin{aligned} \lim_{h \rightarrow 0} \int_a^x \frac{f(x+h) - f(x)}{h} dx \\ &\equiv \lim_{h \rightarrow 0} \left[ \frac{\int_a^{x+h} f(x) dx - \int_a^x f(x) dx}{h} - \frac{\int_a^{a+h} f(x) dx - \int_a^a f(x) dx}{h} \right] \\ &\equiv \lim_{h \rightarrow 0} \left[ \frac{\int_x^{x+h} f(x) dx}{h} - \frac{\int_a^{a+h} f(x) dx}{h} \right]. \end{aligned}$$

\* Ibid., § 249.

† Ibid., § 246.

‡ Ibid., § 250.

§ The upper and lower right-hand derivatives of a function  $F(x)$  are denoted by  $\Lambda F$  and  $\lambda F$  respectively. Vallée Poussin proves (Cours d'Analyse, § 115) by means of elementary properties of derivatives that if the set  $E$  where a derivate of a continuous function has the same sign is of measure zero, the derivate becomes infinite in a part of  $E$ . But more follows from Vallée Poussin's proof than his statement implies, and it is this stronger result that is stated in (E).

|| See Lebesgue, Leçons sur l'Intégration, p. 120.

Applying the mean value theorem to the integrals on the right, we have

$$\lim_{h=0} \int_a^x \frac{f(x+h) - f(x)}{h} dx = \lim_{h=0} [f(x + \vartheta_1 h) - f(a + \vartheta_2 h)],$$

where  $0 < \vartheta_1 < 1$ ,  $0 < \vartheta_2 < 1$  in  $(a, b)$ . Since  $f(x)$  is continuous,

$$\lim_{h=0} \int_a^x \frac{f(x+h) - f(x)}{h} dx = f(x) - f(a),$$

uniformly.

**2. The function  $M(x)$ .** Consider a function, which we shall name the *measure function*, defined as follows:

Let  $\delta_i$  be any set of non-overlapping intervals (and therefore a denumerable set of intervals) in the interval  $(a, b)$  and let  $M(x)$  denote the sum of the intervals  $\delta_i$  or parts thereof to the left of  $x$ . We may write

$M(x) = \sum_a^x \delta_i$ . Important properties of  $M(x)$  are included in the following

**THEOREM.** *The function  $M(x)$ , defined for all real values of  $x$ , is absolutely continuous, monotone increasing, and always less than or equal to  $\sum_1^\infty \delta_i$ . It has a derivative  $M'(x)$  for all  $x$ 's except for a nul set.*

To show that  $M(x)$  is absolutely continuous it need be noted merely that its variation over any set of intervals  $\eta_i$  is  $\leq \Sigma \eta_i$ . It is evident that  $M(x) \leq \sum_1^\infty \delta_i$ , and also that it is monotone increasing, that is,  $\Delta M$  and  $\lambda M$  are always positive or zero.

Denoting by  $\Delta$  the set of points covered by the  $\delta$ 's and by  $C\Delta$  the set complementary to  $\Delta$  with respect to  $(a, b)$ , it is evident that  $M(x)$  has a derivative  $M'(x) = 1$  for inner points of  $\Delta$ . We shall show that  $M'(x) = 0$  for all points of  $C\Delta$  except for a set of measure zero.

By the preceding lemma and (B),

$$M(b) = \lim_{h=0} \int_\Delta \frac{M(t+h) - M(t)}{h} dt + \lim_{h=0} \int_{C\Delta} \frac{M(t+h) - M(t)}{h} dt,$$

from which we have

$$\lim_{h=0} \int_{C\Delta} \frac{M(t+h) - M(t)}{h} dt = 0,$$

since  $M'(x) = 1$  for any inner point of the  $\delta$ 's. Since  $\Delta M$  and  $\lambda M$  are always positive or zero, this shows that  $\lambda M = 0$  in  $C\Delta$  except for a nul set, and therefore  $M(x) = \int_a^x \lambda M(t) dt$ .

Similarly,

$$\begin{aligned} \text{meas } C\Delta &= \int_a^b \lambda(t - M(t))dt = \int_a^b (1 - \Lambda M)dt \\ &= \int_{\Delta} (1 - \Lambda M)dt + \int_{C\Delta} (1 - \Lambda M)dt \\ &= \int_{C\Delta} (1 - \Lambda M)dt \\ &= \text{meas } C\Delta - \int_{C\Delta} \Lambda M dt. \end{aligned}$$

Thus  $\int_{C\Delta} \Lambda M dt = 0$ . Hence  $\lambda M = \Lambda M = 0$  in  $C\Delta$  except for a nul set, which completes the theorem.

This theorem is a special case of Lebesgue's general theorem that any continuous function of limited variation in an interval has a derivative at all points of that interval except for a nul set, but the proof of the theorem for this measure function is much simpler than that for the general function of limited variation. It is upon the proof for the special case that we shall base a proof for the general case.

**3. THEOREM I.** *A function  $f(x)$  continuous in an interval and with a bounded derivate has a derivative almost everywhere in that interval.\**

Let  $K$  be the upper bound of  $|\Delta f|$  in the interval, and note that if one derivate of  $f(x)$  is bounded all of them are bounded.† From the Theorem on Convergence (A), it follows that if  $\epsilon$  and  $\delta$  are positive and arbitrarily small, writing  $\Delta f = f(x + h) - f(x)$ ,  $\Delta f + \epsilon > \Delta f/h$  except for a set  $E_1$  of measure  $< \delta$  and  $\Delta f/h + \epsilon > \lambda f$  except for a set  $E_2$  of measure  $< \delta$ , if  $h$  is small enough. Now enclose  $E_1$  and  $E_2$  in sets of open intervals  $\delta_i$  and  $\delta'_i$ , form the two measure functions  $M_{E_1}(x)$  and  $M_{E_2}(x)$ ‡ and write

$$\varphi_1(x) = KM_{E_1}(x), \quad \varphi_2(x) = KM_{E_2}(x).$$

Then the functions

$$\Psi_1(x) = f(x) + \epsilon x + \varphi_1(x) - \int_a^x \frac{\Delta f}{h} dx$$

and

$$\Psi_2(x) = \int_a^x \frac{\Delta f}{h} dx + \epsilon x + \varphi_2(x) - f(x)$$

\* "Almost everywhere in an interval" means "except for a set of measure zero in that interval."

† This fact is due to Dini. See Vallée Poussin, Cours d'Analyse, § 112, II.

‡ These functions are determinate only when the sets of intervals  $\delta_i$  and  $\delta'_i$ , have been chosen.

are each always monotone increasing, since

$$\Delta\Psi_1(x) = \Delta f + \epsilon + \varphi_1' - \frac{\Delta f}{h} > 0,$$

$$\Delta\Psi_2(x) = \frac{\Delta f}{h} + \epsilon + \varphi_2' - \lambda f > 0.$$

So that

$$\lambda\Psi_1(x) = \lambda f + \epsilon + \varphi_1' - \frac{\Delta f}{h} > 0,$$

$$\lambda\Psi_2(x) = \frac{\Delta f}{h} + \epsilon + \varphi_2' - \Delta f > 0.$$

Hence

$$\frac{\Delta f}{h} - \varphi_1' - \epsilon < \Delta f < \frac{\Delta f}{h} + \varphi_2' + \epsilon,$$

$$\frac{\Delta f}{h} - \varphi_1' - \epsilon < \lambda f < \frac{\Delta f}{h} + \varphi_2' + \epsilon.$$

Thus, except for a set  $E_1 + E_2$  of measure  $< 2\delta$ ,  $\Delta f$  and  $\lambda f$  differ from  $\Delta f/h$  by a quantity  $\leq 2\epsilon$ , which shows that  $\Delta f = \lambda f$  except for a nul set, which proves the theorem.

**THEOREM II.** *The indefinite integral of a function measurable and bounded has that function as a derivative almost everywhere.*

Let  $f(x)$  be measurable and bounded in  $(a, x)$ , and let

$$F(x) - F(a) = \int_a^x f(x)dx. \quad (1)$$

It is to be noted that  $F(x)$  is absolutely continuous and hence measurable.

Since  $f(x)$  is bounded, the Theorem of the Mean (C) is applicable and we have

$$\min f(x) \leq \frac{F(x+h) - F(x)}{h} = \int_x^{x+h} \frac{f(x)}{h} dx \leq \max f(x).$$

That is,  $\Delta F$  is bounded, and therefore, by Theorem I,  $F'(x)$  exists almost everywhere.

According to the preceding lemma

$$F(x) - F(a) = \lim_{h=0} \int_a^x \frac{F(x+h) - F(x)}{h} dx,$$

the Riemann integral of the lemma being replaced by a Lebesgue integral since the integrand is bounded and measurable. Then by Lebesgue's Theorem (D) relative to the passage to the limit under the integral sign, and by Theorem I,

$$F(x) - F(a) = \int_a^x F'(x)dx. \quad (2)$$

From (1) and (2) it follows that  $\int_a^x [F'(x) - f(x)]dx = 0$ , and also  $\int_i [F'(x) - f(x)]dx = 0$ , where  $i$  is any interval or finite set of intervals in  $(a, x)$ . Then  $F'(x) = f(x)$  almost everywhere, for suppose  $F'(x) - f(x) > k$ ,  $k$  positive, over a set  $E$  of measure  $\delta$ . Enclose  $E$  in a finite number of intervals  $\alpha_i$  the sum of whose lengths differ from  $\delta$  by as little as you please. Then  $\int_i [F'(x) - f(x)]dx > k\delta$ , which is a contradiction. The supposition  $f(x) - F'(x) > k$  leads to a contradiction similarly, and the theorem is proved.

The truncation process of Vallée Poussin can be used to deal with the case where the integrand is not bounded. We prove

**THEOREM III.** *If a continuous function  $F(x)$  is non-decreasing in an interval  $(a, x)$ , any one, as  $\Lambda$ , of its derivatives is summable and*

$$\int_a^x \Lambda F dx \leq F(x) - F(a).$$

Let  $E$  be the set of points where  $\Lambda F \leq n$ , a positive number, and let  $CE$  denote the set complementary to  $E$  with respect to the interval  $(a, x)$ . Let  $\Lambda_n = \Lambda F$  in  $E$ , and  $\Lambda_n = n$  in  $CE$ . Then  $\Lambda_n$  is bounded and measurable in  $(a, x)$  and hence summable. Let  $\Phi_n(x) = \int_a^x \Lambda_n dx$ . By Theorem II,  $\Phi_n'(x) = \Lambda_n$  almost everywhere. Then  $\Phi_n'(x) = \Lambda F$  except for a nul set of  $E$  and the set  $CE$ . By the Theorem of the Mean (C), for points of  $CE$ ,

$$n \leq \frac{\Phi_n(x+h) - \Phi_n(x)}{h} = \frac{\int_x^{x+h} \Lambda_n dx}{h} \leq \Delta F,$$

that is,  $\Lambda F - \Phi_n'(x) \geq 0$  everywhere in  $CE$ . Then  $\Lambda F - \Phi_n'(x) < 0$  only in a nul set of  $E$ . But in  $E$  this difference is always bounded, and hence by Theorem (E) it is never negative, and  $\Lambda F \geq \Phi_n'$  always. That is

$$\int_a^x \Lambda_n dx \leq F(x) - F(a).$$

Let  $n$  become infinite. Since the left-hand member is bounded by the right-hand member, it approaches a finite limit, which by Definition II is

$\int_a^x \Lambda F dx$ . That is,

$$\int_a^x \Lambda F dx \leq F(x) - F(a).$$

4. **FUNDAMENTAL THEOREM IV.** *A function  $F(x)$  continuous and of limited variation in an interval  $(a, x)$  has any one of its derivatives, as  $\Lambda$ , summable in that interval.*

Since  $F(x)$  is continuous and of limited variation  $F(x) = F_1(x) - F_2(x)$  where  $F_1(x)$  and  $F_2(x)$  are continuous and non-decreasing. Since  $\Lambda F = \Lambda(F_1 - F_2) \cong \Lambda F_1 - \Lambda F_2$ , and  $\Lambda(F_1 + F_2) \leq \Lambda F_1 + \Lambda F_2$ , we may write

$$\Lambda F_1 + \Lambda F_2 \cong \Lambda(F_1 + F_2) \cong \Lambda(F_1 - F_2) \cong \Lambda F_1 - \Lambda F_2.$$

The derivatives  $\Lambda F_1$  and  $\Lambda F_2$  are summable (Theorem III) and, in view of the remarks on Definitions II and III, it follows that  $\Lambda F = \Lambda(F_1 - F_2)$  is summable.

5. In conclusion, we prove

**THEOREM V.** *The integral of the derivative of a function absolutely continuous in an interval is the variation of the function in that interval.*

Let  $f(x)$  be a function absolutely continuous in the interval  $(a, x)$ . Since  $f(x)$  is absolutely continuous, it is of limited variation. Then  $\Delta f$  is summable (Theorem IV) and hence the measure of the set  $E$  of points where  $\Delta f \geq M$ ,  $M$  a number large at pleasure, is small with  $1/M$ . Let the points  $E$  be enclosed in a set of non-overlapping intervals  $\delta_i$  where  $\Sigma \delta_i$  is small with  $1/M$ . Let  $C\delta$  denote the set complementary to  $\Sigma \delta_i$  with respect to  $(a, x)$ .

By the lemma above

$$\begin{aligned} f(x) - f(a) &= \lim_{h=0} \int_a^x \frac{f(t+h) - f(t)}{h} dt \\ &= \lim_{h=0} \int_{C\delta} \frac{f(t+h) - f(t)}{h} dt + \lim_{h=0} \int_{\Sigma \delta_i} \frac{f(t+h) - f(t)}{h} dt. \end{aligned}$$

Since over  $C\delta$  we have

$$\left| \frac{f(t+h) - f(t)}{h} \right| < M,$$

the first limit in the last member of the preceding equation is found by (D) to be  $\int_{C\delta} f'(t) dt$ . The last integral equals  $\Sigma_s [f(x_{i+1}) - f(x_i)]$  which, since  $f(x)$  is absolutely continuous, approaches zero with  $\Sigma \delta_i$ . That is, if  $f(x)$  is absolutely continuous,

$$f(x) - f(a) = \int_a^x f'(t) dt.$$

Since the integral of any summable function is absolutely continuous, we have

**VITALI'S THEOREM.** *A necessary and sufficient condition that a function be the indefinite integral of a derivative is that it be absolutely continuous.*